7.8 Parameter Error Estimates

The original paper by Avni *et al.* is a helpful reference here, as it gives the likelihood function and indeed the solution to this problem! But it does leave out a few details. The main point of this example is to illustrate how difficult analysis can become from the likelihood point of view, even for a simple thing like a mean.

We will work with twice the negative log of the likelihood function, which we call S. In the notation of Avni *et al.*, we have

$$S = -2 \sum_{k=1}^{M} N_k \log f_k - 2 \sum_{k=1}^{M} U_k \log \sum_{k'=k}^{M} f_{k'}.$$

The double summation arises because of the upper limits and is responsible for all the subsequent algebraic problems.

We have in addition a normalization constraint

$$\sum_{k=1}^{M} f_k = 1$$

and we also want to fix the mean to some particular value m

$$\sum_{k=1}^{M} k f_k = m.$$

We are just working with bin numbers rather than some physical scale.

Now maximum likelihood corresponds to a minimum value of S, but to meet the constraints we use the standard technique of multiplying the constraint equations by Lagrange multipliers μ and λ. Using the prefix δ to denote a small variation in the following variable, we get by the usual rules of differentiation:

$$\mu \sum_{k=1}^{M} \delta f_k = 0, \tag{1}$$

$$\lambda \sum_{k=1}^{M} k\, \delta f_k = 0 \tag{2}$$

and

$$\sum_{k=1}^{M} N_k \frac{\delta f_k}{f_k} + \sum_{k=1}^{M} U_k \frac{\sum_{k'=k}^{M} \delta f_{k'}}{\sum_{k'=k}^{M} f_{k'}}.$$

writing out the last term in this equation carefully, it is a double sum

$$\sum_{k=1}^{M} U_k \frac{(\delta f_k + \delta f_{k+1} + \ldots + \delta f_M)}{\sum_{k'=k}^{M} f_{k'}}$$

and rearranging, we get something that begins to look like the answer we are looking for

$$\sum_{k=1}^{M}\left(\frac{N_k}{f_k}+\sum_{k'=1}^{M}\frac{U_{k'}}{\sum_{k''=k'}^{M} f_{k''}}\right)\delta f_k = 0.$$

By the usual reasoning, since we have the extra freedom of the Lagrange multipliers, we can choose all the δf_k to be zero except one, giving (for each k)

$$\frac{N_k}{f_k}+\sum_{k'=1}^{M}\frac{U_{k'}}{\sum_{k''=k'}^{M} f_{k''}}-\mu-k\lambda=0 \tag{3}$$

after subtracting the variation of the two constraint equations. This gives something that is clearly related to the answer

$$f_k=\frac{N_k}{\mu+k\lambda-\sum_{k'=1}^{M}\frac{U_{k'}}{\sum_{k''=k'}^{M} f_{k''}}}$$

but now we have to find λ and μ.
Suppose for the moment that there is no constraint on the mean, so λ vanishes. We can find μ from rearranging and applying the normalization constraint, Equation 1. The first step is

$$\sum_{k=1}^{M}\mu f_k - f_k\sum_{k'=1}^{M}\frac{U_{k'}}{\sum_{k''=k'}^{M} f_{k''}}=\sum_{k=1}^{M}N_k$$

and we note that $\sum_k f_k = 1$ and $\sum_k N_k = \tilde{J}$ where $\tilde{J}$ is the number of detected objects. Expanding out the remaining term carefully, it comes to just $J-\tilde{J}$ if J is the total number of objects. It follows that $\mu = J$ and we have the Avni *et al.* result.

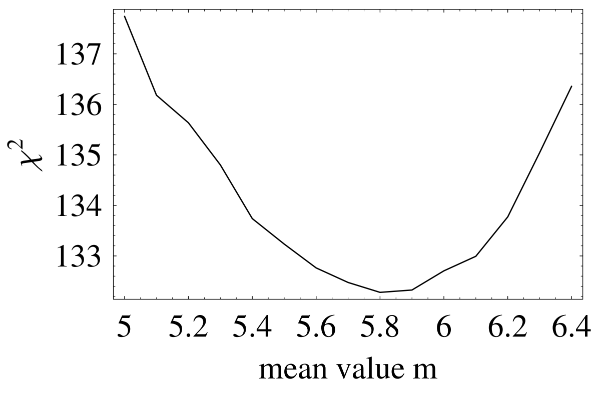

Figure 1: χ^2 plotted against mean value m.

Alas, if the constraint on the mean applies there doesn't seem to be a way of simplifying things. The Lagrange approach however does produce a convenient form, Equation 3.

Combining this with the constraints, Equations 1 and 2, we can solve simultaneously (albeit numerically) for the f_k and the μ and λ. One way of doing this is just to minimize the sum of squares:

$$\sum_{k=1}^{M}\left(\frac{N_k}{f_k}+\sum_{k'=1}^{M}\frac{U_{k'}}{\sum_{k''=k'}^{M} f_{k''}}-\mu-k\lambda\right)^2+\left(1-\sum_{k=1}^{M} f_k\right)^2+\left(m-\sum_{k=1}^{M} k f_k\right)^2.$$

This is brutal, but it works easily for small numbers of bins M. The result can be plugged back into S and we can find out how it varies as a function of the assumed mean m.
A Bayesian solution is easier – if we can use the asymptotic Gaussian form of the likelihood, we get the posterior distribution for the f_k and getting the distribution for the sum is not too bad if the covariance matrix is nearly diagonal (as it often seems to be in these sorts of problems). Working this through is an interesting extra exercise, although it does require quite a lot of differentiation!